What is a positive geometry?

1 Polynomial Equalities and Inequalities

In what follows, take k to be an algebraically-closed field and $d > 0$ be a positive integer.

1. A subset Y of k^d is **algebraic** if there exist polynomials $S \subseteq k[x_1, \ldots, x_n]$ such that

$$Y = \{y \in k^d \mid f(y) = 0 \text{ for all } f \in S\}.$$

   By contrast, Y is a **(closed) semialgebraic set** if there exist polynomials $S, T \subseteq k[x_1, \ldots, x_n]$ such that

$$Y = \{y \in k^d \mid f(y) = 0 \text{ for all } f \in S\} \cup \{y \in k^d \mid f(y) \geq 0 \text{ for all } f \in T\}.$$

2. Consider the space k^d and for each $x, y \in (k^d \setminus \{0\})$ say

$$x \sim y \qquad \Leftrightarrow \qquad x = \lambda y \text{ for some } \lambda \in k \setminus \{0\}.$$

   This defines an equivalence relation on k^d and we call $k\mathbb{P}^{d-1} = (k^d \setminus \{0\})/\sim$ the **projective space** of dimension $d-1$. Here's a quick explanation of why we use $d-1$: choose represenatives of each equivalence class whose norm is 1, so that knowing $d-1$ coordinates uniquely determines that dth. This means that our projective space only depends on $d-1$ coordinates.

3. Let S be a collection of homogeneous [1] polynomials in $k[z_1, \ldots, z_d]$. The zero set of S is an **projective variety**. That is

$$Z(S) = \{x \in \mathbb{P}^{d-1} \mid f(x) = 0 \text{ for all } f \in S\}.$$

   If $Z(S)$ cannot be described as a union of two algebraic varieties in a nontrivial way, then it is a **irreducible projective variety**. There is a more general object, called an *algebraic variety*. Positive geometries are defined for algebraic varieties, but our examples will all be projective, so we will restrict to that setting.

4. Given a projective variety X defined over k, let $k' \subseteq k$ be a subfield of k. If X is defined by polynomials S whose coefficients lie in k', then we can define

$$X(k') = \{x \in X \mid \text{the coordinates of } x \text{ are in } k'\}.$$

   We will be interested in the setting where $k = \mathbb{C}$ and $k' = \mathbb{R}$.

   (a) Let k be a field containing $\mathbb{R}$ and suppose that X can be described as the vanishing set of $S \subseteq \mathbb{R}[x_1, \ldots, x_d]$.

   (b) Let $S, T \subseteq \mathbb{R}[x_1, \ldots, x_n]$. The **(closed) projective semialgebraic set** defined by S and T is the collection of $\sim$ equivalence classes $\tilde{x}$ such that there exists $x \in \tilde{x}$ such that x is in the (closed) semialgebraic set defined by S and T.

[1] Homogenaity is important! If we want to evaluate f on points of $(k^d \setminus \{0\})/\sim$, we don't want the vanishing to depend on our specific equivalence class representative. If f is homogeneous of degree ℓ, then $f(\lambda z_1, \ldots, \lambda z_d) = \lambda^\ell f(z_1, \ldots, z_d)$. This also illustrates why we care about where f is zero, instead of f evaluating to some other number: if $f(z_1, \ldots, z_d)$ is nonzero, then that nonzero number depends on our represenative!

Example 1. Let $X = \mathbb{CP}^2$, $X(\mathbb{R}) = \mathbb{RP}^2$, and $X_{\geq 0}$ be the projective semialgebraic defined by $x_1 \geq 0$. We claim that $X_{\geq 0} = X(\mathbb{R})$. To see this take any $y = (y_1 : y_2 : y_3) \in X(\mathbb{R})$.

- If $y_1 \neq 0$, then $y \sim \frac{1}{y_1} y$ has first coordinate 1.
- On the other hand, if $y_1 = 0$, then $y \in X_{\geq 0}$ directly.

Example 2. Let $X = \mathbb{CP}^2$, $X(\mathbb{R}) = \mathbb{RP}^2$, and $X_{\geq 0}$ be the projective semialgebraic defined by $x_1 - 1 \geq 0$. Take any $y = (y_1 : y_2 : y_3) \in X(\mathbb{R})$.

- If $y_1 \neq 0$, then $y \sim \frac{2}{y_1} y$ has first coordinate 2.
- On the other hand, if $y_1 = 0$, then there is no λ for which λy has first coordinate nonzero.

Then $X_{\geq 0}$ is the collection of points with $y_1 \neq 0$.

Example 3. Let $X = \mathbb{CP}^2$ and $X(\mathbb{R}) = \mathbb{RP}^2$. Let $X_{\geq 0}$ be the projective semialgebraic defined by

$$\begin{aligned} x_1 - 1 &= 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \\ -3x_2 + x_3 + 3 &\geq 0 \\ 2x_2 - x_3 + 1 &\geq 0 \end{aligned}$$

We check that $(1 : 1 : 1)$ satisfies all of these equations. The first three are trivial. For the last two, we have

$$\begin{aligned} -3(1) + (1) + 3 = 1 \geq 0 \\ 2(1) - (1) + 1 = 2 \geq 0\,. \end{aligned}$$

Note that $(-1 : -1 : -1)$ also satisfies these inequalities, since $(-1 : -1 : -1) \sim (1 : 1 : 1)$.

5. A subset X of $k\mathbb{P}^{d-1}$ is **Zariski closed** if there exist homogeneous polynomials $S \subseteq k[x_1, \ldots, x_n]$ such that

$$X = \{x \in k\mathbb{P}^{d-1} \mid f(x) = 0 \text{ for all } f \in S\}\,.$$

Example 4. The set of $x \in \mathbb{CP}^2$ with $(x_1 - 1)(x_2 - 1) = 0$ and $x_1 x_2 + x_3^2 = 0$ is Zariski closed. This contains $(1 : -9 : 3)$ and $(-8 : 1 : 2)$.

Example 5. Let $v^{(1)}, v^{(2)}, \ldots, v^{(m)} \in \mathbb{R}^d$. The (real) convex hull of these points is

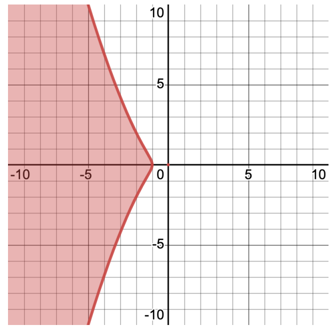
$$\mathsf{conv}_{\mathbb{R}}(v^{(1)}, v^{(2)}, \ldots, v^{(m)}) = \left\{ \sum_{i=1}^{m} \lambda_i \, v^{(i)} \in \mathbb{R}^d \mid \lambda_i \geq 0 \text{ for } i = 1, \ldots, m, \text{ and } \sum \lambda_i = 1 \right\}.$$

A **Euclidean convex polytope** is a subset of $\mathbb{R}^d$ which can be realized as the convex hull of finitely-many points. To obtain a **projective convex polytope**[2] from a Euclidean one, prepend a 1 to each coordinate and considering the corresponding equivalence classes in $\mathbb{RP}^d$. For example, consider the polygon with vertices $(1, 0), (0, 1), (0, 0)$, and $(2, 3)$. The vertices of its projectivization are $(1 : 1 : 0), (1 : 0 : 1), (1 : 0 : 0)$, and $(1 : 2 : 3)$ (imagine embedding the polytope at height one in one dimension higher, then considering all lines through that embedded polytope). We can check that $(1, 1)$ is inside the original polytope. For the projective version, we also have

$$(1 : 2 : 3) + (1 : 0 : 1) + 2 \cdot (1 : 1 : 0) = (4 : 4 : 4) \sim (1 : 1 : 1)\,.$$

[2]Projective polytopes are more general than this. We'll hear more about this later in the quarter!

Non-example 6 (From Nick Addington)**.** Let $X = \mathbb{C}^2$ (this is technically an *affine variety*, but I hope the example is still helpful!). Then $X(\mathbb{R}) = \mathbb{R}^2$. A semialgebraic subset of $\mathbb{R}^2$ is the set of solutions to $-(y^2 + x^2 + x^3) \geq 0$. Here is a picture (created in Desmos):



It's a bit hard to see in the picture, but it is the shaded region, together with the origin. The interior is the shaded region minus its boundary. The closure does not contain the origin, so taking the strict inequality defines and (open) semialgebraic set $X_{>0}$ whose closure is not the closed semialgebraic set defined by those same inequalities.

2 Differentials

Here, let f be a complex function such that f or $\frac{1}{f}$ is differentiable at every point in our domain ("meromorphic").

1. A **pole** of f is a value z such that $\frac{1}{f(z)} = 0$. A pole z_0 is **simple** if there is an integer n such that $(z - z_0)^n f(z)$ is differentiable and nonzero near z_0.

   Example 7. The function $f(z) = \frac{1}{(3-z)^2}$ has a simple pole at 3 of order 2.

2. Now let ω be a differential form with simple pole $z = 0$ (we might want to do a change of variables to get this like $(z - c)^n \mapsto z$). Then we can write ω as a sum of: the part where $z = 0$ is a pole (of order 1) plus the part were $z = 0$ is not a pole. That is, if ω is a differential form in the variables $z, x_1, x_2, \ldots, x_m$, their dx_i's, and dz, then

   $$\omega(z, x_1, x_2, \ldots, x_m) = \alpha(z, x_1, x_2, \ldots, x_m) \wedge \frac{dz}{z} + \beta(z, x_1, x_2, \ldots, x_m)\,.$$

   Since α has no pole at $z = 0$, we can do the Taylor expansion of $\alpha(z, x_1, x_2, \ldots, x_m)$ near $z = 0$. In the limit as z goes to zero, everything in this Taylor expansion goes to zero except the things without z's in them, so locally α looks like a form $\tilde{\alpha}(x_1, x_2, \ldots, x_m)$ defined only in terms of $x_1, \ldots, x_m$. The residue of $\omega(z, x_1, x_2, \ldots, x_m)$ at $z = 0$ is $\tilde{\alpha}(x_1, x_2, \ldots, x_m)$. That is

   $$\mathsf{Res}(\omega, c) = \tilde{\alpha}(x_1, x_2, \ldots, x_m)\,.$$

   Example 8. The differential form $\omega(z) = \frac{dz}{(z-3)} + \frac{dz}{(z-4)}$ has a simple pole at 3 of order 1. After the change of variables $(z - 3) \mapsto z$, we have

   $$\omega(z) = 1 \wedge \frac{dz}{z} + \frac{dz}{(z-1)}\,,$$

   so that the residue at $z = 0$ is 1.

Example 9. The differential form $\omega(x,y) = \frac{y\cos(x)}{(x-3)}\, dxdy$ has a simple pole at 3 of order 1. After the change of variables $(x-3) \mapsto x$, we have

$$\omega(z) = y\cos(x+3)dy \wedge \frac{dx}{x}\,,$$

Expanding $\cos(x+3)$ near zero gives

$$\cos(x+3) = \cos(3) - x\sin(3) - 1/2x^2\cos(3) + 1/6x^3\sin(3) + 1/24x^4\cos(3) - \cdots\,.$$

As x approaches 0, everything goes to zero except the constant term so that the residue at $z = 0$ is $y\cos(3)$.

3 Positive Geometries

Let X be an irreducible complex projective variety. Let $X_{\geq 0} \subseteq X(\mathbb{R})$ be a closed, semialgebraic subset of $X(\mathbb{R})$ such that the closure of the interior $X_{>0}$ is the whole thing $X_{\geq 0}$. Let $\partial X_{\geq 0} = X_{\geq 0} \setminus X_{>0}$ denote the boundary of $X_{\geq 0}$ in X and let ∂X denote the Zariski closure of $\partial X_{\geq 0}$.

The pair $(X, X_{\geq 0})$ is a **positive geometry** if there exists a unique nonzero rational d-form $\Omega(X, X_{\geq 0})$–called the **canonical form**– on X satisfying the following recursive axioms

1. If $d = 0$, then $X = X_{\geq 0}$ is a point, and we define $\Omega(X, X_{\geq 0}) = \pm 1$, where the sign depends on the orientation of the point.

2. If $d > 0$, then

   (a) Every boundary component $(C, C_{\geq 0})$ of $(X, X_{\geq 0})$ is a positive geometry of dimension $d-1$, and

   (b) There exists a unique nonzero rational d-form $\Omega(X, X_{\geq 0})$ on X with the property that

   $$\mathsf{Res}_C \Omega(X, X_{\geq 0}) = \Omega(C, C_{\geq 0})\,.$$

   along every boundary component C (and ther are no singularities elsewhere).

Example 10. Let $a, b \in \mathbb{R}$ with $a < b$. Take X to be $\mathbb{CP}^2$ and $X_{\geq 0}$ the semialgebraic set defined by "projectivizing" the interval $[a, b]$. That is

$$X_{\geq 0} = \{(1:x) \mid x \in \mathbb{R},\ a \leq x \leq b\}$$

Let's check that

$$\Omega(X, X_{\geq 0}) = \frac{dx}{(x-a)} - \frac{dx}{(x-b)}$$

is a candidate for a canonical form for $(X, X_{\geq 0})$. Here $d = 1$, so we look at the second set of conditions. The boundary components are $x = a$ and $x = b$, which are both points (this is the $d = 0$ case, so they are positive geometries by construction). Now we need to check the recursive condition. On the boundary component $\{x = a\}$, we have

$$\mathsf{Res}_{\{x=a\}} \Omega(X, X_{\geq 0}) = 1\,.$$

On the boundary component $\{x = b\}$, we have

$$\mathsf{Res}_{\{x=b\}} \Omega(X, X_{\geq 0}) = -1\,.$$

Non-example 11. Let R be the closed unit disk. Then $(\mathbb{CP}^2, R)$ is *not* a positive geometry, as we will see next week!